

Transition from dissipative to conservative dynamics in equations of hydrodynamics

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Abstract

We show, by using direct numerical simulations and theory, how, by increasing the order of dissipation (α) in equations of hydrodynamics, there is a transition from a dissipative to a conservative system. This remarkable result, already conjectured for the asymptotic case $\alpha \rightarrow \infty$ [U. Frisch et al., Phys. Rev. Lett. **101**, 144501 (2008)], is now shown to be true for any large, but finite, value of α greater than a crossover value $\alpha_{\text{crossover}}$. We thus provide a self-consistent picture of how dissipative systems, under certain conditions, start behaving like conservative systems and hence elucidate the subtle connection between equilibrium statistical mechanics and out-of-equilibrium turbulent flows.

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Since the pioneering work of E. Hopf [1] and T. D. Lee [2], over 60 years ago, physicists have tried to understand the strongly out-of-equilibrium, dissipative turbulent flows by using the tools of classical equilibrium statistical mechanics. What makes such attempts particularly difficult is that although, from a microscopic point of view, fluid motion can be modelled via a Hamiltonian formulation, with statistically steady states governed by an invariant Gibbs measure, a self-consistent macroscopic approach inevitably leads to a dissipative hydrodynamical description with an irreversible energy loss through heat dissipation at the molecular level. In the last few years, however, significant work has gone into our understanding of the interplay between equilibrium statistical mechanics and turbulent flows [3–6, 10]. In particular, the thermalised solutions to the Galerkin-truncated equations of hydrodynamics, such as the three- (3D) or two-dimensional (2D) Euler [3, 7], Gross-Pitaevskii [8] and magnetohydrodynamic [9] equations and the one-dimensional (1D) Burgers equation, have been studied extensively by several authors [5, 10]. For example, it is possible to obtain a conservative dynamical system, which obeys Gibbsian statistical mechanics, for hydrodynamical equations of an ideal fluid where only a finite number of Fourier modes are retained by using the method of Galerkin truncation [3, 10]. Indeed since the first prediction of such thermalised states [11], its existence was shown by Cichowlas et al. [3], for the incompressible, truncated 3D Euler equations, and the explanation of how thermalisation sets in such systems was given by Ray et al. [10] through the phenomenon of *tygers*.

Much of the work discussed above for thermalised states was done for finite-dimensional, conservative systems obeying a Liouville theorem. Therefore it is important to ask if there are connections between such states and dissipative, turbulent flows described by viscous Navier–Stokes-like equations. A partial answer was given by Frisch et al. [12], where the energy spectrum bottleneck, a bump in the spectrum between the inertial and dissipation ranges, in solutions of the incompressible 3D Navier-Stokes and the compressible 1D Burgers equation was attributed to an aborted thermalisation. By using direct numerical simulations (DNSs) and Eddy-Damped-Quasi-Normal-Markovian (EDQNM) calculations [13], it was shown that if we replace the usual viscous operator $\nu \nabla^2 \mathbf{u}$ by the hyperviscous operator $-\nu(-\nabla^2)^\alpha \mathbf{u}$, where ν is the coefficient of viscosity, α is the order of hyperviscosity (dissipativity), and \mathbf{u} the velocity field, the bottleneck becomes stronger with increasing α . The authors observed that for extremely large values of $\alpha \geq 500$, the bottleneck is due to partial

thermalisation observed in [3]. This lead to the intriguing conjecture that the usual bottlenecks, seen in solutions of the Navier–Stokes equation [14–17] and in experiments [14, 18] is possibly because of aborted thermalisation.

The large α limit [12] is extremely important from the point of view of our understanding of hydrodynamical equations. However in most DNSs, which seek to increase the effective inertial range via hyperviscosity, much smaller values of $\alpha \leq 16$ are typically used, which, nevertheless produce significant bottlenecks. For the ordinary Navier–Stokes equation ($\alpha = 1$), a theoretical understanding of the bottleneck was proposed in [19]. Recently, a more complete explanation of this effect was given in [20] where it was shown that this bottleneck has its origins in oscillations in the velocity correlation function. This mechanism is, *apparently*, very different to the aborted thermalisation for large α proposed in [12].

Can this *apparent paradox*, when going from small to large values of α , be resolved? In this paper we show how, by increasing the order of dissipativity (α), we can crossover from one regime [12] to another [20] and thus resolve the paradox. More importantly, as we explain below, our work shows how the tuning of a single parameter α can change a dissipative system to one which displays features of a conservative, Hamiltonian system, leading to a thermalised state. This remarkable result was already conjectured in [12] for the asymptotic case $\alpha \rightarrow \infty$; in this paper we show, by using both DNSs and theory, that this is already the case for a large, but finite, value of α . We thus provide a self-consistent picture of how dissipative systems can start behaving like conservative systems and thus elucidate the subtle connection between equilibrium statistical mechanics and out-of-equilibrium turbulent flows.

The Burgers equation has had a long history of being a testing ground for such ideas related to fluid dynamics [21], and, more recently the chaotic behaviour in conservative systems [10]. Therefore, we begin, with the 1D, unforced, hyperviscous Burgers equation (HBE) :

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = -\nu \left(-\frac{1}{k_d^2} \frac{\partial^2}{\partial x^2} \right)^\alpha u, \quad (1)$$

where, u is the velocity field, x and t , the space and time variables, respectively; ν is the coefficient of kinematic hyperviscosity, and k_d a reference wavenumber. In the limit of vanishing viscosity $\nu \rightarrow 0$, with $\alpha \geq 2$, the solution to Eq. (1) develop oscillations in the boundary layer around the shock; these oscillations result in a bottleneck in the Fourier space energy spectrum [20]. These oscillations – which have been studied by using boundary-layer-expansion techniques [20] – are localised in the neighbourhood of the shock

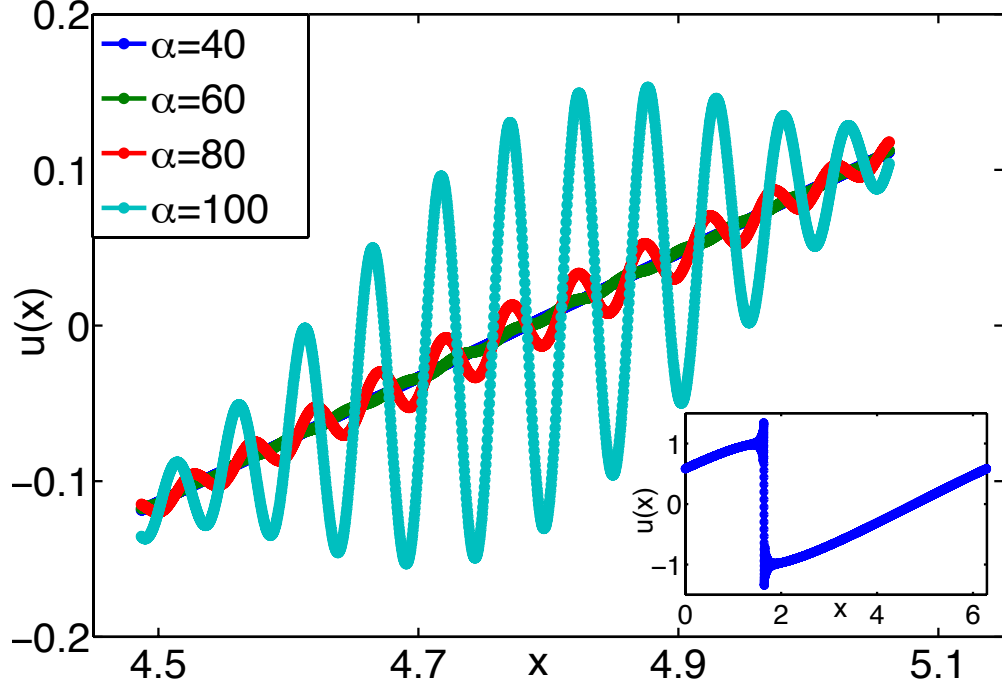


FIG. 1. (color online) Solutions of the HBE, zoomed around x_s , for various values of α (see legend) at $t = 1.5$ showing oscillations at x_s with increasing amplitude as α increases (see text). Inset : Solution of the HBE with no oscillations at x_s for small $\alpha = 20$.

and decay exponentially as one moves away. The wavelength $\lambda_\alpha^{\text{th}}$ and the decay rate K_α^{th} of these oscillations are given by

$$\lambda_\alpha^{\text{th}} = 2\pi\nu^\beta k_d^{-2\alpha\beta} [2^\beta \sin[(2n_* + 1)\beta\pi]]^{-1} \quad (2)$$

$$K_\alpha^{\text{th}} = 2^\beta \nu^{-\beta} k_d^{2\alpha\beta} \cos[(2n_* + 1)\beta\pi]; \quad (3)$$

where, $\beta = \frac{1}{2\alpha-1}$ and n_* is an integer, $0 \leq n_* \leq 2\alpha - 2$, whose value is obtained via linearisation and boundary layer analysis [20].

For extremely large values of $\alpha \geq 500$, the solution of Eq. (1) start thermalising [12] and, at very large times, becomes indistinguishable from the solution $v(x, t)$ of the associated Galerkin-truncated (inviscid), *conservative*, Burgers equation [10] : $\frac{\partial v}{\partial t} + P_{K_G} \frac{1}{2} \frac{\partial v^2}{\partial x} = 0$, where the Galerkin projector P_{K_G} is a low-pass filter which sets to zero all Fourier components with wavenumbers $|k| > k_G$

At this stage, it behooves us to ask the question what happens for intermediate values of α ? And, furthermore, is there a single mechanism which can self-consistently describe the transition from a turbulence regime to a thermalised state in equations of hydrodynamics?

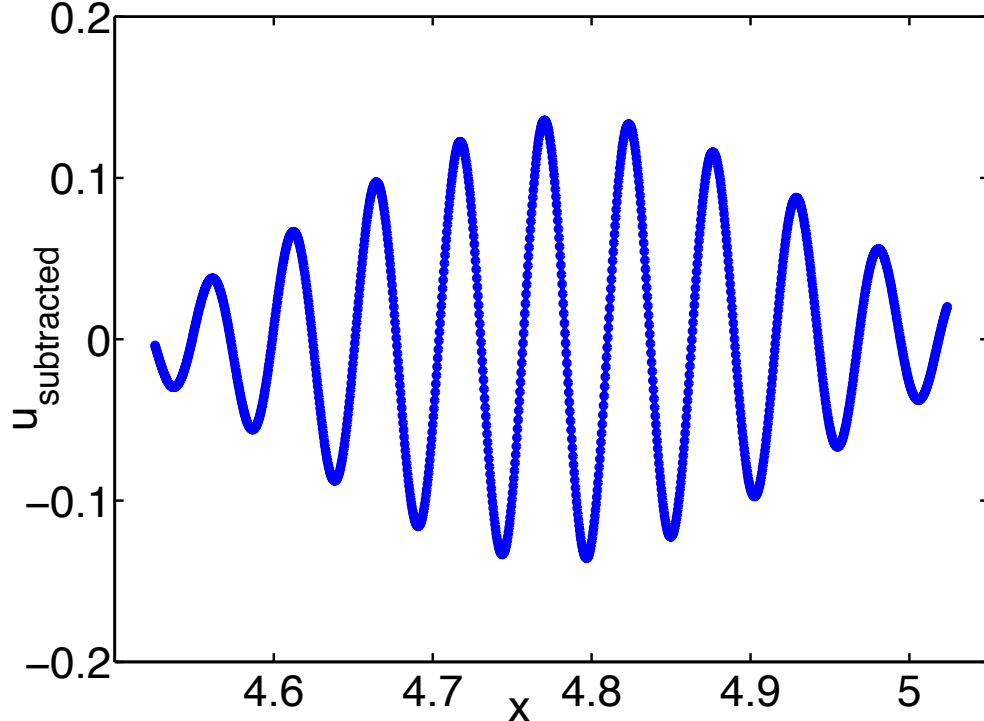


FIG. 2. (color online) The solution of the HBE for $\alpha = 100$, with the solution for the ordinary Burgers subtracted out, zoomed around x_s . A clear symmetric bulge, similar to those seen in inviscid, conservative truncated system [10], is seen.

The onset of thermalisation, in inviscid, finite-dimensional systems of the Euler or the Burgers equation, is due to the birth of structures called tygers. These are caused [10] by the motion of fluid particles interacting resonantly with the waves generated, because of truncation, by small-scale features, such as shocks. The special points in physical space where tygers appear are the so-called *stagnation points* x_s , which, in the case of the Galerkin-truncated Burgers equation, are points which have the same velocity as the shock(s) and a positive local gradient. Is there another way, apart from truncation waves in inviscid systems, for waves to be generated at the stagnation points in a fluid for similar resonant interactions leading to an onset of thermalisation? We will show that for α greater than a crossover value $\alpha_{\text{crossover}}$, a significant fraction of the oscillations, governed by Eqs. (2) and (3), which start from the boundary layer near the shock, must reach x_s and trigger tyger-like structures leading to thermalisation.

In order to answer these questions we first perform pseudo-spectral DNSs of Eq. (1) on a 2π periodic line, with a second-order Runge–Kutta scheme for time-integration. We

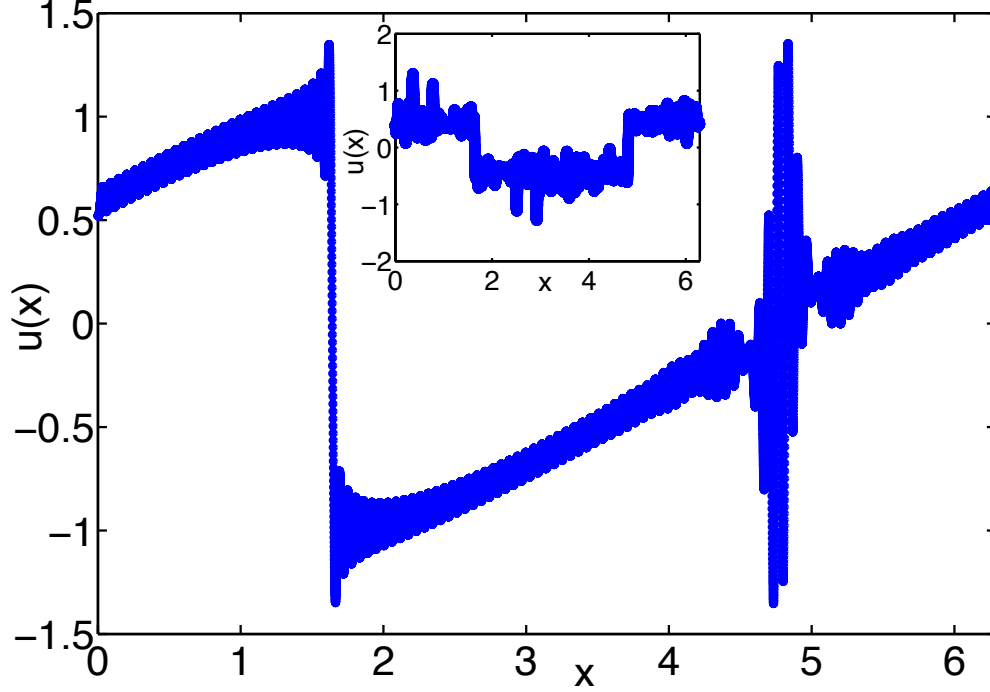


FIG. 3. (color online) Solution of the HBE, for $\alpha = 250$ at $t = 1.5$, showing the presence of a *tyger* [10] at x_s . Inset : $u(x)$ for $\alpha = 250$ at a later time ($t = 5.0$) confirming that for $\alpha \gtrsim \alpha_{\text{crossover}}$ the system eventually thermalises.

use a time step $\delta t = 10^{-4}$, the number of collocation points $N = 16384$, $\nu = 10^{-20}$ and $k_d = 100$. Crucially, we use $2 \leq \alpha \leq 500$ to study this intriguing transition from dissipative dynamics to conservative, thermalised states. Our initial condition $u_0(x) = \sin(x+1.5)$ leads to $x_s = 2\pi - 1.5 \approx 4.8$ and, in the absence of viscosity, shock formation at time $t_* = 1.0$.

We begin our simulations from $\alpha = 2$ and observe [20] that with increasing α , oscillations in a thin layer around the shock become pronounced with a related bottleneck in the energy spectrum. However, near the stagnation point x_s , no oscillations are seen for $\alpha \lesssim 40$. This is clearly seen in a plot of $u(x)$ versus x , as shown in the inset of Fig. 1, at $t = 1.5$ for $\alpha = 20$. However, as α increases, finite, but small, oscillations start to reach x_s from the boundary layer around the shock. This is shown in Fig. 1 for values of $\alpha = 40, 60, 80$, and 100. Furthermore, for values of $\alpha \gtrsim 80$, a distinct bulge, reminiscent of the tigers found in solutions of the Galerkin truncated equation [10], is clearly seen at x_s . This, then, is the first evidence of what triggers thermalisation in a dissipative system and whose dramatic consequences were studied in Ref. [12] for the special case of $\alpha \rightarrow \infty$.

How similar is this bulge, in the dissipative HBE, at x_s for $\alpha \gtrsim 80$, to that seen at t_* for the Hamiltonian system of the Galerkin truncated Burgers equation? In order to answer this question, it is useful to examine the bulge, via $u_{\text{subtracted}} = u - U$, where U is the (non-oscillatory) solution of the inviscid Burgers equation. In Fig. 2 we show this subtracted bulge for $\alpha = 100$ and find that this bulge has the same shape as tygers [10] and is also symmetric around x_s . The wavelength of these oscillations, as is expected from a resonance build-up argument, is the same as the wavelength of the oscillations emanating from the boundary layer around the shock (2). A significant difference between the bulge observed for moderate values of α (Figs. (2) and (1)), and that of a tiger [10], is its large width and the surprisingly small number of oscillations inside it. In the truncated system, the bulge width is proportional to $k_G^{-1/3}$ and the wavelength of the oscillations proportional to $1/k_G$; this yields the number of oscillations in the bulge to be proportional to $k_G^{2/3}$. In the present problem, the width of the bulge is explained as follows : At time t , such that $\tau = t - t_*$, and when the bulge is still symmetric around x_s , resonant interactions are confined to particles such that $\tau \Delta u \equiv \tau |u - u_s| \lesssim \lambda_\alpha^{\text{th}}$, where u_s is the velocity of the shock. We have chosen t such that $\tau \sim 1$, leading to $\Delta u \sim \lambda_\alpha^{\text{th}}$. Given that around x_s the velocity is proportional to x , this yields a bulge width $\sim \lambda_\alpha^{\text{th}}$ with a few oscillations inside.

In the case of the inviscid, Galerkin truncated Burgers equation, the early bulge (tygers) become asymmetric in time, leading to an eventual collapse and thermalised states. In the present dissipative problem, although for reasonably small values of α , a bulge is guaranteed to form at x_s , its eventual dynamics – and indeed whether the system actually thermalises – depends on the interplay between the local dissipation around x_s , the fraction of oscillations reaching x_s from the boundary layer, and the effect of the nonlinearity. For smaller values of α , when the dissipation is strong and the amplitude of oscillations is small, this bulge at large times, remains stationary in time without ever collapsing and leading to a complete thermalisation. However, as α increases, the amplitude of oscillations reaching the stagnation point is significant : Consequently for values of α higher than a threshold $\alpha_{\text{crossover}}$, the local dissipation can no longer compensate for the resonant pile up at x_s leading to the emergence of thermalised states in a manner exactly similar to that of the inviscid truncated systems. Heuristically, an estimate of $\alpha_{\text{crossover}}$ can be obtained as follows : The fraction of amplitude at the boundary layer that reaches x_s is given by $e^{-K_\alpha^{\text{th}} \pi}$. We assume that a significant level of oscillations is present at x_s when at least a fraction $1/e$ of the oscillations produced near

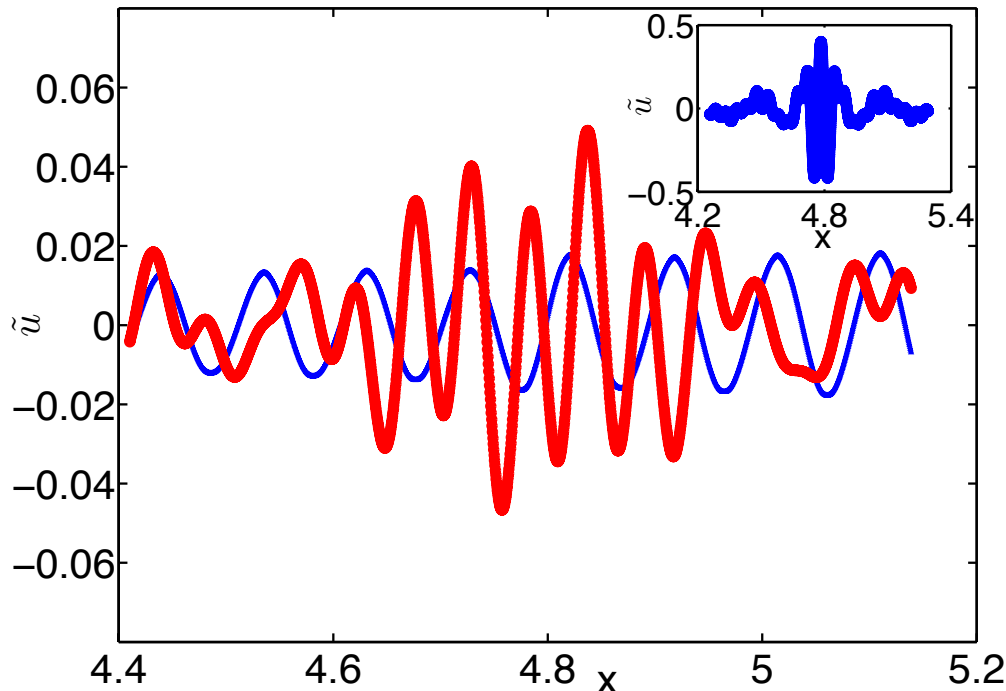


FIG. 4. (color online) The solution \tilde{u} of the linearised equation (5) with initial conditions I1 at time $t = 2.0$ (blue curve) and $t = 2.5$ (red curve) for $\alpha = 100$ (see text). Inset : \tilde{u} with initial conditions I2 at time $t = 10.0$.

the shock reaches x_s , i.e., $K_\alpha^{th}\pi = 1$. By using (3), and in the limit of large α , we obtain $\alpha_{\text{crossover}} = \frac{1}{2}(1 + 50\pi^2) \simeq 230$.

We now examine the accuracy of our estimate of $\alpha_{\text{crossover}}$ through detailed simulations with increasing values of α . As we increase α , our simulations show that the bulge at the x_s reaches a stationary state without collapsing. However at around $\alpha \gtrsim 220$ we observe that the bulge which forms, due to resonance, at x_s , collapses in a finite time and then the system thermalises in a manner reminiscent to the dynamics of the Galerkin truncated Burgers equation [10]. This is best seen in Fig. (3) where we show the solution of the HBE for $\alpha = 250$ at time $t = 1.5$. We note that, just as in the inviscid, Galerkin truncated Burgers equation [10], the bulge at the resonance point becomes very large, assymetric and non-monochromatic with secondary structures on either side of it. This is exactly similar to the onset of thermalisation in conservative systems [10]. Indeed at larger times the solution completely thermalises (inset of Fig. (3), at $t = 5.0$). Our simulations illustrate quite clearly that (a) the heuristic estimate of $\alpha_{\text{crossover}}$ is correct and, more importantly, (b) dissipative

systems, such as the HBE, can thermalise at finite values of the order of dissipativity in a manner similar to that of conserved, truncated systems. The fact that dissipative systems can start to mimic truncated, Hamiltonian system through the tuning of a single parameter (α) is a striking result and resolves a long standing paradox in the area of turbulence and statistical mechanics. It is not hard to conjecture that this cross-over should be possible for $\alpha \rightarrow \infty$ [12]; however, remarkably, we now show that the onset to thermalisation actually occurs at a finite value $\alpha_{\text{crossover}}$.

Let us finally address the question of whether this phenomenon can be captured within a systematic theoretical framework. Rewriting Eq. (1) in terms of the solution U of the inviscid Burgers equation and the discrepancy $\tilde{u} \equiv u - U$, and using $\frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial U^2}{\partial x} = 0$, we obtain,

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x}(U\tilde{u}) + \frac{1}{2} \frac{\partial \tilde{u}^2}{\partial x} = -\nu \left(-\frac{1}{k_d^2} \frac{\partial^2}{\partial x^2} \right)^\alpha (\tilde{u} + U). \quad (4)$$

At times close to t_* , and away from the shock, the discrepancy between the solution of the HBE (1) and the solution of the inviscid Burgers equation is small ($\tilde{u}/U \ll 1$); hence we can drop the quadratic term. Next, we note that U is linear in the spatial variable x away from the shock which implies that higher derivatives of U vanish around x_s . By using these two approximations, we finally obtain the following, analytically more tractable, linear equation :

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial x}(U\tilde{u}) = -\nu \left(-\frac{1}{k_d^2} \frac{\partial^2}{\partial x^2} \right)^\alpha \tilde{u}. \quad (5)$$

We first validate our linear theory by numerically solving (5) for \tilde{u} with a further approximation that U is the solution at t_* of the inviscid Burgers equation, with the initial condition $\sin(x + 1.5)$. We choose two kinds of initial conditions $\tilde{u}_0 = \tilde{u}(t = 0)$: (I1) \tilde{u}_0 is a low amplitude sinusoidal function with a wavenumber equal to 10; and (I2) $\tilde{u}_0 = e^{-K_\alpha^{th}|x-x_{\text{shock}}|} \sin \frac{2\pi(x-x_{\text{shock}})}{\lambda_\alpha^{th}}$, where x_{shock} is the position of the shock. Our numerical integration of Eq. (5) for both classes of initial conditions yield similar results as illustrated in Fig. 4 where we present a representative plot of \tilde{u} , solved for Eq. (5), at time $t = 2$ (blue curve), and, $t = 2.5$ (red curve) for $\alpha = 100$ by using the initial conditions I2; the inset shows the solution of Eq. (5) for initial conditions I1 at time $t = 10.0$. A symmetric bulge at the stagnation point, just like in the solutions Eq. (1) for large α is clearly seen. The essential features of the bulge, i.e., its locality and the fact that it forms at the stagnation point is reproduced by our linear model. Having established the validity of the linear

model to predict the location and the nature of the bulge, we can now solve Eq. (5) by various standard analytical means such as by using the method of separation of variables or through a Fourier transform of Eq. (5), to obtain solutions of Eq. (5) (upto constants) which show the existence of symmetric bulges at x_s which decay on either side of the stagnation point. We should note in passing, that although the linear model predicts well the early stages of the formation of the bulge at x_s , our extensive simulations of the linear model, for various large values of α , not surprisingly, fails capture the collapse of the bulge and eventual thermalisation [10]. A plausible conjecture for this is that the non-linearity, however weak, is responsible for the stretching of the bulge and generating an associated Reynolds stress which must be present to make the symmetric bulge collapse and trigger complete thermalisation.

For the past many decades, a vexing and open question in the areas of turbulence and statistical mechanics is how meaningful are thermalised states in such problems. In this paper we answer this question via detailed numerical simulations and linear models. Our results show that just as in the case of Hamiltonian systems of the Galerkin-truncated equation, where monochromatic truncation waves can reach x_s , leading to an accumulation, via resonance, and eventual thermalisation, similarly, for dissipative systems such as the HBE, for moderately large α , monochromatic boundary layer oscillations reach and accumulate, via the same resonant effect, at x_s . These bulges are the seeds of an eventual thermalised regime and for $\alpha \gtrsim \alpha_{\text{crossover}}$ the dissipative system does thermalise at large times in a manner similar to the inviscid truncated system. Our work thus connects the apparently disconnected worlds of conservative and dissipative systems. Although we have confined ourselves to the one-dimensional Burgers equation, the central result obtained in this paper should be valid in the multidimensional Navier–Stokes equation for the reasons outlined in Refs. [10, 12, 20]. A detailed study of this is beyond the scope of the present paper and is left for the future.

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